Miniversal deformations of pairs of skew-symmetric forms

Andrii Dmytryshyn

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Abstract

We give a miniversal deformation of each pair of skew-symmetric matrices (A,B) under congruence; that is, a normal form with minimal number of independent parameters to which all matrix pairs $(A+E,B+E')$ close to (A,B) can be reduced by congruence transformation $(A+E,B+E')\mapsto\mathcal{S}(E,E')^T(A+E,B+E')\mathcal{S}(E,E'),\mathcal{S}(0,0)=I$, in which $\mathcal{S}(E,E')$ smoothly depends on the entries of E and E'.

1 Introduction

This is a joint work with Vyacheslav Futorny and Vladimir V. Sergeichuk.

The most known and studied in many courses of linear algebra canonical matrices are the Jordan matrices and the canonical matrices for symmetric and hermitian forms and for isometric and selfadjoint operators on unitary and Euclidean spaces. Their generalizations are usually obtained by different methods and often are very intricate. V.I. Arnold [1] pointed out that the reduction of a matrix to its Jordan form is an unstable operation: both the Jordan form and the reduction transformation depend discontinuously on the elements of the original matrix. Therefore, if the elements of a matrix are known only approximately, then it is unwise to reduce the matrix to its Jordan form. Furthermore, when investigating a family of matrices smoothly depending on parameters, then although each individual matrix can be reduced to a Jordan form, it is unwise to do so since in such an operation the smoothness relative to the parameters is lost.

V. I. Arnold obtained a miniversal deformation of Jordan matrix, i.e. a simplest possible normal form, to which not only a given matrix A, but an arbitrary family of matrices close to it can be reduced by means of a similarity transformation smoothly depending on the elements of A in a neighborhood of zero. The problem is useful for applications, when the matrices arise as a result of measures, i.e. their entries are known with errors.

(Mini)versal deformations are known for many kinds of matrices and matrix pencils [4].

Outline

In Section 2 we present the main result in terms of holomorphic functions, and in terms of miniversal deformations. We use the canonical matrices of a pair of skew-symmetric forms given by Thompson [6].

Section 3 is a proof of the main result. The method of constructing deformations is presented and using it we calculate deformations step by step: for the diagonal blocks, for the off diagonal blocks that correspond to the canonical summands of the same type, and for the off diagonal blocks that correspond to the canonical summands of different types.

In Section 4 the constructive proof of the versality of deformations is given.

2 The main theorem

In this section we formulate theorems about miniversal deformations of pairs of skew-symmetric matrices under congruence (it will be proved in the next section), but first we recall a canonical form of pairs of skew-symmetric matrices under congruence.

Define the $n \times n$ matrices

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \qquad I_n := \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix},$$

and the $n \times (n+1)$ matrices

$$F_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad G_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}.$$

The following lemma was proved in [6].

Lemma 2.1. *Every pair of skew-symmetric complex matrices is congruent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form*

$$H_n(\lambda) := \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \right), \quad \lambda \in \mathbb{C}, \tag{1}$$

$$K_n := \left(\begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right), \tag{2}$$

$$L_n := \left(\begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \right). \tag{3}$$

2.1 The main theorem in terms of holomorphic functions

Let (A,B) be a given pair of $n \times n$ skew-symmetric matrices. For all pairs of skew-symmetric matrices $(A+E, B+E')$ that are close to (A,B), we give their normal form $\mathcal{A}(E,E')$ with respect to congruence transformation

$$(A+E, B+E') \mapsto \mathcal{S}(E,E')^T (A+E, B+E') \mathcal{S}(E,E'), \tag{4}$$

in which $\mathcal{S}(E,E')$ is holomorphic at 0 (i.e., its entries are power series in the entries of E and E' that are convergent in a neighborhood of 0) and $\mathcal{S}(0,0)$ is a nonsingular matrix.

Since $\mathcal{A}(0,0) = \mathcal{S}(0,0)^T(A,B)\mathcal{S}(0,0)$, we can take $\mathcal{A}(0,0)$ equal to the congruence canonical form $(A,B)_{\text{can}}$ of (A,B). Then

$$\mathcal{A}(E,E') = (A,B)_{\text{can}} + \mathcal{D}(E,E'), \tag{5}$$

where $\mathcal{D}(E,E')$ is a pair of matrices that is holomorphic at 0 and $\mathcal{D}(0,0) = (0,0)$. In the next theorem we obtain $\mathcal{D}(E,E')$ with the minimal number of nonzero entries that can be attained by using transformation (4).

We use the following notation, in which every star denotes a function of the entries of E and E' that is holomorphic at zero:

• 0_{mn} is the $m \times n$ zero matrix;

• 0_{mn*} is the $m \times n$ matrix $\begin{bmatrix} & & & 0 \\ & 0_{m-1,n-1} & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & * \end{bmatrix}$;

• $0^{\nwarrow}_{mn}$ is the $m \times n$ matrix

$$\begin{bmatrix} * & \\ * & \\ \vdots & 0_{m,n-1} \\ * & \end{bmatrix} \quad \text{if } m \leqslant n \text{ and, respectively, } \begin{bmatrix} * & * & \dots & * \\ & & 0_{m-1,n} & \end{bmatrix} \quad \text{if } m \geqslant n \qquad (6)$$

(if $m = n$, then we can take any of the matrices defined in (6));

• $0^{\nearrow}$, $0^{\searrow}$ and $0^{\swarrow}$ are matrices that are obtained from $0^{\nwarrow}$, by the clockwise rotation through 90°, 180° and 270°, respectively;

• $0^{\leftarrow}_{mn}$ is the $m \times n$ matrix $\begin{bmatrix} * & \\ \vdots & 0_{m,n-1} \\ * & \end{bmatrix}$;

• $0^{\rightarrow}_{mn}$ is the $m \times n$ matrix $\begin{bmatrix} & * \\ 0_{m,n-1} & \vdots \\ & * \end{bmatrix}$;

• $0^{\urcorner}_{mn}$ is the $m \times n$ matrix

$$\begin{bmatrix} * & \dots & * \\ & 0_{m-1,n-1} & \vdots \\ & & * \end{bmatrix} \text{ or } \begin{bmatrix} * & & \\ \vdots & 0_{m-1,n-1} & \\ * & \dots & * \end{bmatrix};$$

• $\mathcal{Q}_{nm}$ with $n < m$ is the $n \times m$ matrix

$$\begin{bmatrix} 0 & \dots & 0 & & & & \\ \vdots & & \vdots & & 0 & & \\ 0 & \dots & 0 & * & \dots & * & 0 \end{bmatrix} \qquad (m - n \text{ stars});$$

when $n \geq m$ than $\mathcal{Q}_{nm} = 0$.

Further, we will usually omit the indices m and n.

Our main result is the following theorem, which we reformulate in a more abstract form in Theorem 2.2.

Theorem 2.1. *Let*

$$(A,B)_{\mathrm{can}} = X_1 \oplus \dots \oplus X_t \tag{7}$$

be a canonical pair of skew-symmetric complex matrices for congruence, in which $X_1,\dots,X_t$ are pairs of the form (1)–(3). *Its simplest miniversal deformation can be taken in the form $(A,B)_{\mathrm{can}} + \mathcal{D}$ in which $\mathcal{D}$ is a $(0,*)$ matrix pair (the stars denote independent parameters, up to skew-symmetry, see Remark 2.1) whose matrices are partitioned into blocks conformally to the decomposition* (7)*:*

$$\mathcal{D} = \left(\begin{bmatrix} \mathcal{D}_{11} & \dots & \mathcal{D}_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{t1} & \dots & \mathcal{D}_{tt} \end{bmatrix}, \begin{bmatrix} \mathcal{D}'_{11} & \dots & \mathcal{D}'_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{D}'_{t1} & \dots & \mathcal{D}'_{tt} \end{bmatrix} \right) \tag{8}$$

These blocks are defined as follows. Write

$$\mathcal{D}(X_i) := (\mathcal{D}_{ii}, \mathcal{D}'_{ii}) \tag{9}$$

$$\mathcal{D}(X_i, X_j) := ((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (\mathcal{D}_{ji}, \mathcal{D}'_{ji})) \quad \text{if } i < j, \tag{10}$$

(Remaind that $((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (\mathcal{D}_{ji}, \mathcal{D}'_{ji})) = ((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (-\mathcal{D}^T_{ij}, -\mathcal{D}'^T_{ij}))$, thus we drop the second pair in the notation.)

then

(i) *The diagonal blocks of $\mathcal{D}$ are defined by*

$$\mathcal{D}(H_n(\lambda)) = \left(0, \begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}\right) \tag{11}$$

$$\mathcal{D}(K_t) = \left(\begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}, 0\right) \tag{12}$$

$$\mathcal{D}(L_m) = (0,0). \tag{13}$$

(ii) *The off-diagonal blocks of $\mathcal{D}$ whose horizontal and vertical strips contain summands of $(A,B)_{\mathrm{can}}$ of the same type are defined by*

$$\mathcal{D}(H_n(\lambda),\, H_m(\mu)) = \begin{cases} (0,\ 0) & \text{if } \lambda \neq \mu \\ \left(0, \begin{bmatrix} 0^{\searrow} & 0^{\swarrow} \\ 0^{\nearrow} & 0^{\nwarrow} \end{bmatrix}\right) & \text{if } \lambda = \mu \end{cases} \tag{14}$$

$$\mathcal{D}(K_n, K_m) = \left(\begin{bmatrix} 0^{\searrow} & 0^{\swarrow} \\ 0^{\nearrow} & 0^{\nwarrow} \end{bmatrix}, 0\right) \tag{15}$$

$$\mathcal{D}(L_n, L_m) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0_* \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{Q}^T \\ \mathcal{Q} & 0^\urcorner \end{bmatrix} \right). \tag{16}$$

(iii) *The off-diagonal blocks of $\mathcal{D}$ whose horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types are defined by:*

$$\mathcal{D}(H_n(\lambda), K_m) = (0, 0) \tag{17}$$

$$\mathcal{D}(H_n(\lambda), L_m) = \left(0, \begin{bmatrix} 0 & 0^\leftarrow \end{bmatrix}\right) \tag{18}$$

$$\mathcal{D}(K_n, L_m) = (0^\rightarrow, 0). \tag{19}$$

Remark 2.1 (About independency of parameters). A matrix pair $\mathcal{D}$ is skew-symmetric. It means that each of $\mathcal{D}_{ij}, i < j$ and each of $\mathcal{D}'_{ij}, i < j$ contain independent parameters and just one half of parameters of $\mathcal{D}_{ii}$ and $\mathcal{D}'_{ii}$ are independent (i.e. all parameters in the upper triangular parts of matrices of $\mathcal{D}$ are independent).

The matrix pair $\mathcal{D}$ from Theorem 2.1 will be constructed in Section 3 as follows. The vector space

$$V := \{C^T(A,B)_{\text{can}} + (A,B)_{\text{can}}C \,|\, C \in \mathbb{C}^{n \times n}\}$$

is the tangent space to the congruence class of $(A,B)_{\text{can}}$ at the point $(A,B)_{\text{can}}$ since

$$(I + \varepsilon C)^T (A,B)_{\text{can}} (I + \varepsilon C) = (A,B)_{\text{can}} + \varepsilon (C^T(A,B)_{\text{can}} + (A,B)_{\text{can}}C)$$
$$+\varepsilon^2 C^T (A,B)_{\text{can}} C$$

for all n-by-n matrices C and each $\varepsilon \in \mathbb{C}$. Then $\mathcal{D}$ satisfies the following condition:

$$\mathbb{C}_c^{n \times n} \times \mathbb{C}_c^{n \times n} = V \oplus \mathcal{D}(\mathbb{C}) \tag{20}$$

in which $\mathbb{C}_c^{n \times n}$ is the space of all $n \times n$ skew-symmetric matrices, $\mathcal{D}(\mathbb{C})$ is the vector space of all matrix pairs obtained from $\mathcal{D}$ by replacing its stars by complex numbers. Thus, one half of the number of stars in $\mathcal{D}$ is equal to the codimension of the congruence class of $(A,B)_{\text{can}}$ (note that the total number of stars is always even). Lemma 3.2 from the next section ensures that any matrix pair with entries 0 and $*$ that satisfies (20) can be taken as $\mathcal{D}$ in Theorem 2.1.

2.2 The main theorem in terms of miniversal deformations

The notion of a miniversal deformation of a matrix with respect similarity was given by V. I. Arnold [1] (see also [3, § 30B]). This notion is easily extended to matrix pairs with respect to congruence.

A *deformation* of a pair of $n \times n$ matrices (A, B) is a holomorphic mapping $\mathcal{A}$ from a neighborhood $\Lambda \subset \mathbb{C}^k$ of $\vec{0} = (0, \dots, 0)$ to the space of pairs of $n \times n$ matrices such that $\mathcal{A}(\vec{0}) = A$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two deformations of (A, B) with the same parameter space $\mathbb{C}^k$. Then $\mathcal{A}$ and $\mathcal{B}$ are considered as *equal* if they coincide on some neighborhood of $\vec{0}$ (this means that each deformation is a germ); $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if the identity matrix I_n possesses a deformation $\mathcal{I}$ such that

$$\mathcal{B}(\vec{\lambda}) = \mathcal{I}(\vec{\lambda})^T \mathcal{A}(\vec{\lambda}) \mathcal{I}(\vec{\lambda}) \tag{21}$$

for all $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ in some neighborhood of $\vec{0}$.

Definition 2.1. A deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a matrix pair (A, B) is called *versal* if every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of (A, B) is equivalent to a deformation of the form $\mathcal{A}(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$, where all $\varphi_i(\vec{\mu})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of (A, B) is called *miniversal* if there is no versal deformation having less than k parameters.

By a $(0, *)$ *matrix pair* we mean a pair $\mathcal{D}$ of matrices whose entries are 0 and $*$. We say that a matrix pair *is of the form* $\mathcal{D}$ if it can be obtained from $\mathcal{D}$ by replacing the stars with complex numbers. Denote by $\mathcal{D}(\mathbb{C})$ the space of all matrix pairs of the form $\mathcal{D}$, and by $\mathcal{D}(\vec{\varepsilon})$ the pair of parametric matrices obtained from $\mathcal{D}$ by replacing each (i, j) star with the parameters ε_{ij} or f_{ij}. This means that

$$\mathcal{D}(\mathbb{C}) := \Big(\bigoplus_{(i,j) \in \mathcal{I}_1(\mathcal{D})} \mathbb{C} E_{ij} \Big) \times \Big(\bigoplus_{(i,j) \in \mathcal{I}_2(\mathcal{D})} \mathbb{C} E_{ij} \Big), \tag{22}$$

$$\mathcal{D}(\vec{\varepsilon}) := \Big(\sum_{(i,j) \in \mathcal{I}_1(\mathcal{D})} \varepsilon_{ij} E_{ij}, \sum_{(i,j) \in \mathcal{I}_2(\mathcal{D})} f_{ij} E_{ij} \Big), \tag{23}$$

where

$$\mathcal{I}_1(\mathcal{D}), \mathcal{I}_2(\mathcal{D}) \subseteq \{1, \dots, n\} \times \{1, \dots, n\} \tag{24}$$

are the sets of indices of the stars in the first and the second matrices, respectively, of the pair $\mathcal{D}$, and E_{ij} is the elementary matrix whose (i,j) entry is 1 and the others are 0.

We say that a miniversal deformation of (A,B) is *simplest* if it has the form $(A,B)+\mathcal{D}(\vec{\varepsilon})$, where $\mathcal{D}$ is a $(0,*)$ matrix pair. If $\mathcal{D}$ has no zero entries, then it defines the deformation

$$\mathcal{U}(\vec{\varepsilon}) := \Big(A + \sum_{i,j=1}^{n} \varepsilon_{ij}E_{ij},\ B + \sum_{i,j=1}^{n} \varepsilon_{ij}E_{ij}\Big). \tag{25}$$

Since each matrix pair is congruent to its canonical matrix pair, it suffices to construct miniversal deformations of canonical matrix pairs (a direct sum of the summands (1)-(3)). These deformations are given in the following theorem, which is a stronger form of Theorem 2.1.

Theorem 2.2. *A simplest miniversal deformation of the canonical matrix pair $(A,B)_{\rm can}$ of skew-symmetric matrices for congruence can be taken in the form $(A,B)_{\rm can}+\mathcal{D}(\vec{\varepsilon})$, where $\mathcal{D}$ is the $(0,*)$ matrix pair partitioned into blocks as in* (8) *that are defined by* (11) - (19) *in the notation* (9) - (10)*.*

3 Proof of the main theorem

3.1 A method of construction of miniversal deformations

Now we give a method of construction of simplest miniversal deformations, which will be used in the proof of Theorem 2.2.

The deformation (25) is universal in the sense that every deformation $\mathcal{B}(\mu_1,\dots,\mu_l)$ of (A,B) has the form $\mathcal{U}(\vec{\varphi}(\mu_1,\dots,\mu_l))$, where $\varphi_{ij}(\mu_1,\dots,\mu_l)$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_{ij}(\vec{0})=0$. Hence every deformation $\mathcal{B}(\mu_1,\dots,\mu_l)$ in Definition 2.1 can be replaced by $\mathcal{U}(\vec{\varepsilon})$, which proves the following lemma.

Lemma 3.1. *The following two conditions are equivalent for any deformation $\mathcal{A}(\lambda_1,\dots,\lambda_k)$ of pair of matrices (A,B):*

(i) *The deformation $\mathcal{A}(\lambda_1,\dots,\lambda_k)$ is versal.*

(ii) *The deformation* (25) *is equivalent to $\mathcal{A}(\varphi_1(\vec{\varepsilon}),\dots,\varphi_k(\vec{\varepsilon}))$ in which all $\varphi_i(\vec{\varepsilon})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0})=0$.*

For a pair of n-by-n skew-symmetric matrices (A,B) we define

$$T(A,B) := \{C^T(A,B) + (A,B)C \mid C \in \mathbb{C}^{n\times n}\}. \tag{26}$$

If U is a subspace of a vector space V, then each set $v+U$ with $v \in V$ is called a *coset of U in V*.

Lemma 3.2. *Let $(A,B) \in (\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ and let $\mathcal{D}$ be a pair of $(0,*)$ matrices of the size $n \times n$. The following are equivalent:*

(i) *The deformation $(A,B) + \mathcal{D}(E,E')$ defined in (22) is miniversal.*

(ii) *The vector space $(\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ decomposes into the direct sum*

$$(\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n}) = T(A,B) \oplus \mathcal{D}(\mathbb{C}). \tag{27}$$

(iii) *Each coset of $T(A,B)$ in $(\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ contains exactly one matrix pair of the form $\mathcal{D}$.*

Proof. Define the action of the group $GL_n(\mathbb{C})$ of nonsingular n-by-n matrices on the space $[\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n}]$ by

$$(A,B)^S = S^T(A,B)S, \qquad (A,B) \in [\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n}], \quad S \in GL_n(\mathbb{C}).$$

The orbit $(A,B)^{GL_n}$ of (A,B) under this action consists of all pairs of skew-symmetric matrices that are congruent to the pair (A,B).

The space $T(A,B)$ is the tangent space to the orbit $(A,B)^{GL_n}$ at the point (A,B) since

$$\begin{aligned}(A,B)^{I+\varepsilon C} &= (I+\varepsilon C)^T(A,B)(I+\varepsilon C)\\ &= (A,B) + \varepsilon(C^T(A,B) + (A,B)C) + \varepsilon^2 C^T(A,B)C\end{aligned}$$

for all n-by-n matrices C and $\varepsilon \in \mathbb{C}$. Hence $\mathcal{D}(\vec{\varepsilon})$ is transversal to the orbit $(A,B)^{GL_n}$ at the point (A,B) if

$$(\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n}) = T(A,B) + \mathcal{D}(\mathbb{C})$$

(see definitions in [3, § 29E]; two subspaces of a vector space are called *transversal* if their sum is equal to the whole space).

This proves the equivalence of (i) and (ii) since a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation [2, Section 1.6]. The equivalence of (ii) and (iii) is obvious. □

In Section 4 we give a constructive proof of the versality of each deformation $(A,B)+\mathcal{D}(\vec{\varepsilon})$ in which $\mathcal{D}$ satisfies (27): we construct a deformation $\mathcal{I}(\vec{\varepsilon})$ of the identity matrix such that $\mathcal{D}(\vec{\varepsilon})=\mathcal{I}(\vec{\varepsilon})^T\mathcal{U}(\vec{\varepsilon})\mathcal{I}(\vec{\varepsilon})$, where $\mathcal{U}(\vec{\varepsilon})$ is defined in (25).

Thus, a simplest miniversal deformation of $(A,B)\in(\mathbb{C}_c^{n\times n},\mathbb{C}_c^{n\times n})$ can be constructed as follows. Let $(T_1,\dots,T_r)$ be a basis of the space $T(A,B)$, and let $(E_1,\dots,E_{\frac{n(n-1)}{2}},F_1,\dots,F_{\frac{n(n-1)}{2}})$ be the basis of $(\mathbb{C}_c^{n\times n},\mathbb{C}_c^{n\times n})$ consisting of all elementary matrices (E_{ij},F_{ij}). Removing from the sequence $(T_1,\dots,T_r,E_1,\dots,E_{\frac{n(n-1)}{2}},F_1,\dots,F_{\frac{n(n-1)}{2}})$ every pair of matrices that is a linear combination of the preceding matrices, we obtain a new basis $(T_1,\dots,T_r,E_{i_1},\dots,E_{i_k},F_{i_1},\dots,F_{i_m})$ of the space $(\mathbb{C}_c^{n\times n},\mathbb{C}_c^{n\times n})$. By Lemma 3.2, the deformation

$$\mathcal{A}(\varepsilon_1,\dots,\varepsilon_k,f_1,\dots,f_m)=(A+\varepsilon_1E_{i_1}+\cdots+\varepsilon_kE_{i_k},B+f_1F_{i_1}+\cdots+f_mF_{i_m})$$

is miniversal.

For each pair of $m\times m$ skew-symmetric matrices (M,N) and each pair $n\times n$ skew-symmetric matrices (L,P), define the vector spaces

$$V(M,N):=\{S^T(M,N)+(M,N)S|S\in\mathbb{C}^{m\times m}\} \tag{28}$$

$$V((M,N),(L,P)):=\{(R^T(L,P)+(M,N)S,S^T(M,N)+(L,P)R)| \quad S\in\mathbb{C}^{m\times n},R\in\mathbb{C}^{n\times m}\} \tag{29}$$

Lemma 3.3. *Let $(A,B)=(A_1,B_1)\oplus\cdots\oplus(A_t,B_t)$ be a block-diagonal matrix in which every (A_i,B_i) is $n_i\times n_i$. Let $\mathcal{D}$ be a pair of $(0,*)$ matrices having the size of (A,B). Partition it into blocks $(\mathcal{D}_{ij},\mathcal{D}'_{ij})$ conformably to the partition of (A,B) (see (8)). Then $(A,B)+\mathcal{D}(E,E')$ is a simplest miniversal deformation of (A,B) for congruence if and only if*

(i) *every coset of $V(A_i,B_i)$ in $(\mathbb{C}_c^{n_i\times n_i},\mathbb{C}_c^{n_i\times n_i})$ contains exactly one matrix of the form $(\mathcal{D}_{ii},\mathcal{D}'_{ii})$, and*

(ii) *every coset of $V((A_i,B_i),(A_j,B_j))$ in $(\mathbb{C}^{n_i\times n_j},\mathbb{C}^{n_i\times n_j})\oplus(\mathbb{C}^{n_j\times n_i},\mathbb{C}^{n_j\times n_i})$ contains exactly two pairs of matrices $((W_1,W_2),(Q_1,Q_2))$ in which (W_1,W_2) is of the form $(\mathcal{D}_{ij},\mathcal{D}'_{ij})$, (Q_1,Q_2) is of the form $(\mathcal{D}_{ji},\mathcal{D}'_{ji})=(-\mathcal{D}_{ij}^T,-\mathcal{D}'^T_{ij})$.*

Proof. By Lemma 3.2(iii), $(A,B)+\mathcal{D}(\vec{\varepsilon})$ is a simplest miniversal deformation of (A,B) if and only if for each $(C,C') \in (\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ the coset $(C,C') + T(A,B)$ contains exactly one (D,D') of the form $\mathcal{D}$; that is, exactly one

$$(D,D') = (C,C') + S^T(A,B) + (A,B)S \in \mathcal{D}(\mathbb{C}) \qquad \text{with } S \in \mathbb{C}^{n\times n}. \tag{30}$$

Partition (D,D'), (C,C'), and S into blocks conformably to the partition of (A,B). By (30), for each i we have $(D_{ii},D'_{ii}) = (C_{ii},C'_{ii}) + S_{ii}^T(A_i,B_i) + (A_i,B_i)S_{ii}$, and for all i and j such that $i<j$ we have

$$\begin{aligned}\left(\begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj}\end{bmatrix}, \begin{bmatrix} D'_{ii} & D'_{ij} \\ D'_{ji} & D'_{jj}\end{bmatrix}\right) &= \left(\begin{bmatrix} C_{ii} & C_{ij} \\ C_{ji} & C_{jj}\end{bmatrix}, \begin{bmatrix} C'_{ii} & C'_{ij} \\ C'_{ji} & C'_{jj}\end{bmatrix}\right) \\ + \begin{bmatrix} S_{ii}^T & S_{ji}^T \\ S_{ij}^T & S_{jj}^T\end{bmatrix}\left(\begin{bmatrix} A_i & 0 \\ 0 & A_j\end{bmatrix}, \begin{bmatrix} B_i & 0 \\ 0 & B_j\end{bmatrix}\right) &+ \left(\begin{bmatrix} A_i & 0 \\ 0 & A_j\end{bmatrix}, \begin{bmatrix} B_i & 0 \\ 0 & B_j\end{bmatrix}\right)\begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj}\end{bmatrix}.\end{aligned} \tag{31}$$

Thus, (30) is equivalent to the conditions

$$(D_{ii},D'_{ii}) = (C_{ii},C'_{ii}) + S_{ii}^T(A_i,B_i) + (A_i,B_i)S_{ii} \in \mathcal{D}_{ii}(\mathbb{C}), (1 \leqslant i \leqslant t) \tag{32}$$

$$\begin{gathered}((D_{ij},D'_{ij}),(D_{ji},D'_{ji})) = ((C_{ij},C'_{ij}),(C_{ji},C'_{ji}))+ \\ ((S_{ji}^TA_j + A_iS_{ij}, S_{ji}^TB_j + B_iS_{ij}),(S_{ij}^TA_i + A_jS_{ji}, S_{ij}^TB_i + B_jS_{ji})) \in \mathcal{D}_{ij}(\mathbb{C}) \oplus \mathcal{D}_{ji}(\mathbb{C}) \\ (1 \leqslant i < j \leqslant t)\end{gathered} \tag{33}$$

Hence for each $(C,C') \in (\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ there exists exactly one $(D,D') \in \mathcal{D}$ of the form (30) if and only if

(i′) for each $(C_{ii},C'_{ii}) \in (\mathbb{C}_c^{n_i\times n_i}, \mathbb{C}_c^{n_i\times n_i})$ there exists exactly one $(D_{ii},D'_{ii}) \in \mathcal{D}_{ii}$ of the form (32), and

(ii′) for each $((C_{ij},C'_{ij}),(C_{ji},C'_{ji})) \in (\mathbb{C}^{n_i\times n_j}, \mathbb{C}^{n_i\times n_j}) \oplus (\mathbb{C}^{n_j\times n_i}, \mathbb{C}^{n_j\times n_i})$ there exists exactly one $((D_{ij},D'_{ij}),(D_{ji},D'_{ji})) \in \mathcal{D}_{ij}(\mathbb{C}) \oplus \mathcal{D}_{ji}(\mathbb{C})$ of the form (33).

This proves the lemma. □

Corollary 3.1. In the notation of Lemma 3.3, $(A,B) + \mathcal{D}(\vec{\varepsilon})$ is a miniversal deformation of (A,B) if and only if each pair of submatrices of the form

$$\left(\begin{bmatrix} A_i + \mathcal{D}_{ii}(\vec{\varepsilon}) & \mathcal{D}_{ij}(\vec{\varepsilon}) \\ \mathcal{D}_{ji}(\vec{\varepsilon}) & A_j + \mathcal{D}_{jj}(\vec{\varepsilon})\end{bmatrix}\begin{bmatrix} B_i + \mathcal{D}'_{ii}(\vec{\varepsilon}) & \mathcal{D}'_{ij}(\vec{\varepsilon}) \\ \mathcal{D}'_{ji}(\vec{\varepsilon}) & B_j + \mathcal{D}'_{jj}(\vec{\varepsilon})\end{bmatrix}\right), \qquad i<j,$$

is a miniversal deformation of the pair $(A_i \oplus A_j, B_i \oplus B_j)$.

Let us start to prove Theorem 2.1. Each X_i in (7) is of the form $H_n(\lambda)$, or K_n, or L_n, and so there are 9 types of pairs $\mathcal{D}(X_i)$ and $\mathcal{D}(X_i, X_j)$ with $i < j$; they are given (11)–(19). It suffices to prove that the pairs (11)–(19) satisfy the conditions (i) and (ii) of Lemma 3.3.

3.2 Diagonal blocks of matrices of $\mathcal{D}$

Fist we verify that the diagonal blocks of $\mathcal{D}$ defined in part (i) of Theorem 2.1 satisfy the condition (i) of Lemma 3.3.

3.2.1 Diagonal blocks $\mathcal{D}(H_n(\lambda))$ and $\mathcal{D}(K_n)$

Firstly we consider the pair of blocks $H_n(\lambda)$. Without loss of generality we can assume that $\lambda = 0$, because for any X we have $-XJ_n(\lambda)^T + J_n(\lambda)X = -X(\lambda I + J_n(0))^T + (\lambda I + J_n(0))X = -XJ_n(0)^T + J_n(0)X$. Hence the deformation of K_n is equal to the deformation of $H_n(\lambda)$ up to the permutation of matrices.

Due to Lemma 3.3(i), it suffices to prove that each pair of skew-symmetric $2n$-by-$2n$ matrices $(A, B) = ([A_{ij}]_{i,j=1}^2, [B_{ij}]_{i,j=1}^2)$ can be reduced to exactly one pair of matrices of the form (11) by adding

$$\begin{aligned}
\Delta(A, B) = (\Delta A, \Delta B) &= \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} \right) \\
&= \begin{bmatrix} S_{11}^T & S_{21}^T \\ S_{12}^T & S_{22}^T \end{bmatrix} \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \right) \\
&\quad + \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \right) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\
&= \left(\begin{bmatrix} S_{21} - S_{21}^T & S_{11}^T + S_{22} \\ -S_{11} - S_{22}^T & S_{12}^T - S_{12} \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} -S_{21}^T J_n(0)^T + J_n(0) S_{21} & S_{11}^T J_n(0) + J_n(0) S_{22} \\ -S_{22}^T J_n(0)^T - J_n(0)^T S_{11} & S_{12}^T J_n(0) - J_n(0)^T S_{12} \end{bmatrix} \right)
\end{aligned} \tag{34}$$

in which $S = [S_{ij}]_{i,j=1}^2$ is an arbitrary $2n$-by-$2n$ matrix. Note that every pair of n-by-n blocks of our pair of matrices is changed independently.

Let us consider the first pair of blocks $(A_{11}, B_{11}) = (S_{21} - S_{21}^T, -S_{21}^T J_n(0)^T + J_n(0)S_{21})$ in which S_{21} is an arbitrary n-by-n matrix. Obviously, that adding $\Delta A_{11} = S_{21} - S_{21}^T$ we reduce A_{11} to zero. To preserve A_{11} we must hereafter

take S_{21} such that $S_{21} - S_{21}^T = 0$. This means that S_{21} is a symmetric matrix. We reduce B_{11} by adding $\Delta B_{11} = -S_{21}^T J_n(0)^T + J_n(0) S_{21}$.

$$\Delta B_{11} = -\begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & s_{22} - s_{13} & s_{23} - s_{14} & \dots & s_{2n} \\ -s_{22} + s_{13} & 0 & s_{33} - s_{24} & \dots & s_{3n} \\ -s_{23} + s_{14} & -s_{33} + s_{24} & 0 & \dots & s_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_{2n} & -s_{3n} & -s_{4n} & \dots & 0 \end{bmatrix} \quad (35)$$

Since our matrix is skew-symmetric we need to reduce just the upper triangular part of it and the lower triangular part will be reduced automatically. An upper half of each skew diagonal of ΔB_{11} has unique variables, thus we reduce upper half of each skew diagonal of B_{11} independently. For the first $n-1$ skew diagonals we have the system of equations with the matrix (the first skew diagonal is zero):

$$\left(\begin{array}{cccccc|c} 1 & -1 & & & & & x_1 \\ & 1 & -1 & & & & x_2 \\ & & \ddots & \ddots & & & \vdots \\ & & & 1 & -1 & & x_k \end{array}\right) \text{where } x_1 \dots x_k \text{ are corresponding elements of } B_{11}; \quad (36)$$

In this paper non-specified entries of matrices of systems of equations are zeros and we denote corresponding elements of the matrix that we reduce by $x_1 \dots x_k$. By the Kronecker-Capelli theorem this system has a solution. Therefore we can reduce each of the first $n-1$ skew-diagonals of A to zero.

For the last n upper parts of skew diagonals we have the system of equa-

tions with the matrix (the last skew diagonal is zero):

$$\begin{pmatrix} 1 & -1 & & & & x_1 \\ & 1 & -1 & & & x_2 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & -1 & x_{k-1} \\ & & & & 1 & x_k \end{pmatrix} \tag{37}$$

By the Kronecker-Capelli theorem this system has a solution too. Therefore we can reduce the last n skew-diagonals of B_{11} to zero. Thus we can reduce B_{11} to zero matrix by adding ΔB_{11}.

The possibility of the reduction (A_{22}, B_{22}) by adding $(S_{12}^T - S_{12}, S_{12}^T J_n(0) - J_n(0)^T S_{12})$ to zero follows almost immediately now.

We have $0 = B_{11} - S_{21}^T J_n(0)^T + J_n(0) S_{21}$ where B_{11} is a skew-symmetric matrix. Multiplying this equality by the n-by-n matrix

$$Z := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \tag{38}$$

from both sides and taking into account that $Z^2 = I$ and $Z J_n(0)^T Z = J_n(0)$ we get

$$0 = Z B_{11} Z - Z S_{21}^T Z J_n(0) + J_n(0)^T Z S_{21} Z.$$

This ensures that the pair of blocks (A_{22}, B_{22}) can be set to zero since $Z B_{11} Z$ and $Z S_{21} Z$ are arbitrary skew-symmetric and symmetric matrices, respectively.

To the pair of blocks (A_{21}, B_{21}) we can add $\Delta(A_{21}, B_{21}) = (S_{11}^T + S_{22}, S_{11}^T J_n(0) + J_n(0) S_{22})$. Adding $S_{11}^T + S_{22}$ we reduce A_{21} to zero. To preserve A_{21} we must hereafter take S_{11} and S_{22} such that $-S_{22} = S_{11}^T$. Thus we

add $\Delta B_{21} = -S_{22}J_n(0) + J_n(0)S_{22}$, with any matrix S_{22}.

$$
\begin{aligned}
\Delta B_{21} = -&\begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix} \\
&= \begin{bmatrix} s_{21} & s_{22} - s_{11} & s_{23} - s_{12} & \dots & s_{2n} - s_{1n-1} \\ s_{31} & s_{32} - s_{21} & s_{33} - s_{22} & \dots & s_{3n} - s_{2n-1} \\ s_{41} & s_{42} - s_{31} & s_{43} - s_{32} & \dots & s_{4n} - s_{3n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -s_{n1} & -s_{n2} & \dots & -s_{nn-1} \end{bmatrix} \qquad (39)
\end{aligned}
$$

Each diagonal of ΔB_{21} has unique variables, thus we examine each of them independently. For the first m diagonals (starting from the lower left corner) we have the system of equations with the matrix

$$
\begin{pmatrix} 1 & & & & x_1 \\ -1 & 1 & & & x_2 \\ & \ddots & \ddots & & \vdots \\ & & -1 & 1 & x_k \\ & & & -1 & x_{k+1} \end{pmatrix} \qquad (40)
$$

The matrix of this system has k columns and $k+1$ rows and its rank is equal to k. But the rank of the full matrix of the system is $k+1$, by the Kronecker-Capelli theorem this matrix does not have a solution. If we turn down the first or the last equation of the system (this means that we do not set the first or the last element of each of m diagonals of our matrix to zero), than it will have a solution.

For the last $m-1$ diagonals we have the system of equations with the matrix

$$
\begin{pmatrix} 1 & -1 & & & x_1 \\ & 1 & -1 & & x_2 \\ & & \ddots & \ddots & \vdots \\ & & & 1 \quad -1 & x_k \end{pmatrix} \qquad (41)
$$

Obviously, it has a solution. Therefore we can set each element of the matrix B_2 to zero except either the first column or the last row.

The block $(-S_{11}-S_{22}^T, -S_{22}^T J_n(0)^T - J_n(0)^T S_{11})$ is analogous to the previous one up to the transposition and the sign.

Collecting together the answers about all blocks we get $\mathcal{D}(H_n(\lambda)) = \left(0, \begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}\right)$ and respectively $\mathcal{D}(K_n) = \left(\begin{bmatrix} 0 & 0^{\swarrow} \\ 0^{\nearrow} & 0 \end{bmatrix}, 0\right)$.

3.2.2 Diagonal blocks $\mathcal{D}(L_n)$

We act as in the previous subsubsection (using Lemma 3.3(i)). We prove that each pair $(A,B) = ([A_{ij}]_{i,j=1}^2, [B_{ij}]_{i,j=1}^2)$ of skew-symmetric $2n+1$-by-$2n+1$ matrices can be set to zero by adding

$$\begin{aligned}
\Delta(A,B) = (\Delta A, \Delta B) &= \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix}\right) \\
&= \begin{bmatrix} S_{11}^T & S_{21}^T \\ S_{12}^T & S_{22}^T \end{bmatrix} \left(\begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix}\right) \\
&+ \left(\begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix}\right) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\
&= \left(\begin{bmatrix} -S_{21}^T F_n^T + F_n S_{21} & S_{11}^T F_n + F_n S_{22} \\ -S_{22}^T F_n^T - F_n^T S_{11} & S_{12}^T F_n - F_n^T S_{12} \end{bmatrix},\right. \\
&\qquad \left.\begin{bmatrix} -S_{21}^T G_n^T + G_n S_{21} & S_{11}^T G_n + G_n S_{22} \\ -S_{22}^T G_n^T - G_n^T S_{11} & S_{12}^T G_n - G_n^T S_{12} \end{bmatrix}\right)
\end{aligned} \tag{42}$$

in which $S = [S_{ij}]_{i,j=1}^2$ is an arbitrary matrix. Each pair of blocks of (A,B) is changed independently.

We can add $\Delta(A_{11}, B_{11}) = (-S_{21}^T F_n^T + F_n S_{21}, -S_{21}^T G_n^T + G_n S_{21})$ in which S_{21} is an arbitrary $n+1$-by-n matrix to the pair of blocks (A_{11}, B_{11}). Obviously, that adding $-S_{21}^T F_n^T + F_n S_{21}$ we reduce A_{11} to zero. To preserve A_{11} we must hereafter take only S_{21} such that $F_n S_{21} = S_{21}^T F_n^T$.

This means that

$$S_{21}^T = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} & s_{1n+1} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} & s_{2n+1} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} & s_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} & s_{nn+1} \end{bmatrix}.$$

The matrix S_{21} without the last row is symmetric. Now we reduce B_{11} by adding

$$\Delta B_{11} = -\begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} & s_{1n+1} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} & s_{2n+1} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} & s_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} & s_{nn+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} \\ s_{1n+1} & s_{2n+1} & s_{3n+1} & \dots & s_{nn+1} \end{bmatrix}$$

$$= \begin{cases} -s_{ij+1} + s_{i+1j} & \text{if} \quad i < j \\ s_{ij+1} - s_{i+1j} & \text{if} \quad i > j \quad \text{where} \quad i, j = 1 \dots n. \\ 0 & \text{if} \quad i = j \end{cases} \tag{43}$$

The upper part of each skew diagonal of ΔB_{11} has unique variables, thus we reduce each diagonal of B_{11} independently. We have the system of equations with the matrix (41) for upper part of each skew diagonal, which has a solution, by the Kronecker-Capelli theorem. It follows that we can reduce every skew-diagonal of B_{11} to zero. Hence we can reduce (A_{11}, B_{11}) to zero.

To the pair of blocks (A_{12}, B_{12}) we can add $\Delta(A_{12}, B_{12}) = (S_{11}^T F_n + F_n S_{22}, S_{11}^T G_n + G_n S_{22})$ in which S_{11} and S_{22} are arbitrary matrices of the corresponding size. Adding $S_{11}^T F_n + F_n S_{22}$ we reduce A_{12} to zero. To preserve A_{12} we must hereafter take only S_{11} and S_{22} such that $F_n S_{22} = -S_{11}^T F_n$. This means, that

$$S_{22} = \begin{bmatrix} & & & 0 \\ & -S_{11}^T & & 0 \\ & & & \vdots \\ & & & 0 \\ y_1 & y_2 & \dots & y_{n+1} \end{bmatrix}.$$

We reduce B_{12} by adding

$$\Delta B_{12} = S_{11}^T G_n + G_n S_{22} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{11} & -s_{12} & -s_{13} & \dots & -s_{1n} & 0 \\ -s_{21} & -s_{22} & -s_{23} & \dots & -s_{2n} & 0 \\ -s_{31} & -s_{32} & -s_{33} & \dots & -s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n1} & -s_{n2} & -s_{n3} & \dots & -s_{nn} & 0 \\ y_1 & y_2 & y_3 & \dots & y_n & y_{n+1} \end{bmatrix}$$

$$= - \begin{bmatrix} s_{21} & -s_{11}+s_{22} & -s_{12}+s_{23} & \dots & -s_{1n-1}+s_{2n} & -s_{1n} \\ s_{31} & -s_{21}+s_{32} & -s_{22}+s_{33} & \dots & -s_{2n-1}+s_{3n} & -s_{2n} \\ s_{41} & -s_{31}+s_{42} & -s_{32}+s_{43} & \dots & -s_{3n-1}+s_{4n} & -s_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n1} & -s_{n-11}+s_{n2} & -s_{n-12}+s_{n3} & \dots & -s_{nn-1}+s_{nn} & -s_{n-1n} \\ -y_1 & -s_{n1}-y_2 & -s_{n2}-y_3 & \dots & -s_{nn+1}-y_n & -s_{nn}-y_{n-1} \end{bmatrix} \tag{44}$$

It is easily seen that we can set B_{12} to zero by adding ΔB_{12} (diagonal by diagonal).

The pair of blocks $(-S_{22}^T F_n^T - F_n^T S_{11}, -S_{22}^T G_n^T - G_n^T S_{11})$ is analogous to the previous one up to the transposition and the sign.

To the pair of blocks (A_{22}, B_{22}) we add $\Delta(A_{22}, B_{22}) = (S_{12}^T F_n - F_n^T S_{12}, S_{12}^T G_n - G_n^T S_{12})$ in which S_{12} is an arbitrary n-by-$n+1$ matrix. Obviously, that adding $S_{12}^T F_n - F_n^T S_{12}$ we reduce A_{22} to zero. To preserve A_{22} we must hereafter take only S_{12} such that $S_{12}^T F_n = F_n^T S_{12}$. This means that

$$S_{12} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} & 0 \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} & 0 \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} & 0 \end{bmatrix}.$$

The matrix S_{12} without the last column is symmetric. Now we reduce B_{22}

by adding

$$\Delta B_{22} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} & 0 \\ s_{12} & s_{22} & s_{23} & \dots & s_{2n} & 0 \\ s_{13} & s_{23} & s_{33} & \dots & s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & s_{nn} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s_{11} & s_{12} & s_{13} & \dots & s_{1n-1} & s_{1n} \\ -s_{11} & 0 & s_{22} - s_{13} & s_{23} - s_{14} & \dots & s_{2n-1} - s_{1n} & s_{2n} \\ -s_{12} & s_{13} - s_{22} & 0 & s_{33} - s_{24} & \dots & s_{3n-1} - s_{2n} & s_{3n} \\ -s_{13} & s_{14} - s_{23} & s_{24} - s_{33} & 0 & \dots & s_{4n-1} - s_{3n} & s_{4n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -s_{1n-1} & s_{1n} - s_{2n-1} & s_{2n} - s_{3n-1} & s_{3n} - s_{4n-1} & \dots & 0 & s_{nn} \\ -s_{1n} & -s_{2n} & -s_{3n} & -s_{4n} & \dots & -s_{nn} & 0 \end{bmatrix}.$$

The upper part of each skew diagonal has unique variables, thus we reduce each of them independently. We have the system of equations (37) which has a solution for the upper part of each skew diagonal. It follows that we can reduce every skew-diagonal of B_{22} to the diagonal with zero elements only. Hence we can reduce (A_{22}, B_{22}) to zero.

Collecting together the answers about all pairs of blocks of this pair we get $\mathcal{D}(L_n) = 0$.

3.3 Off-diagonal blocks of matrices of $\mathcal{D}$ that correspond to summands of $(A, B)_{\text{can}}$ of the same type

Now we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(ii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A, B)_{\text{can}}$ of the same type.

3.3.1 Pairs of blocks $\mathcal{D}(H_n(\lambda), H_m(\mu))$ and $\mathcal{D}(K_n, K_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to exactly one group of the form (14) by adding

$$(R^T H_m(\mu) + H_n(\lambda)S, S^T H_n(\lambda) + H_m(\mu)R), \quad S \in \mathbb{C}^{2n\times 2m}, R \in \mathbb{C}^{2m\times 2n}.$$

Obviously, that we can reduce only on the first pair of matrices, the second pair will be reduced automatically. So we reduce a pair (A,B) of $2n$-by-$2m$ matrices by adding

$$\Delta(A,B) = R^T H_m(\mu) + H_n(\lambda)S =$$

$$(R^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S).$$

It is clear that we can reduce A to zero. To preserve A we must hereafter take only R and S such that

$$R^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0.$$

This means

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} = \begin{bmatrix} -R_{22}^T & R_{12}^T \\ R_{21}^T & -R_{11}^T \end{bmatrix}.$$

$B := \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}$ is reduced by adding

$$\Delta B := \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & J_m(\mu) \\ -J_m(\mu)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T & R_{12}^T \\ R_{21}^T & -R_{11}^T \end{bmatrix}$$

$$= \begin{bmatrix} -R_{21}^T J_m(\mu)^T + J_n(\lambda) R_{21}^T & R_{11}^T J_m(\mu) - J_n(\lambda) R_{11}^T \\ -R_{22}^T J_m(\mu)^T + J_n(\lambda)^T R_{22}^T & R_{12}^T J_m(\mu) - J_n(\lambda)^T R_{12}^T \end{bmatrix} \quad (45)$$

B_{11} is reduced by adding

$$\Delta B_{11} = -R_{21}^T J_m(\mu)^T + J_n(\lambda) R_{21}^T$$

$$= \begin{cases} (\lambda-\mu)r_{ij} + r_{i+1j} - r_{ij+1} & \text{if} \quad 1 \le i \le n-1, \quad 1 \le j \le m-1 \\ (\lambda-\mu)r_{ij} + r_{i+1j} & \text{if} \quad 1 \le i \le n-1, \quad j = m \\ (\lambda-\mu)r_{ij} - r_{ij+1} & \text{if} \quad 1 \le j \le m-1, \quad i = n \\ (\lambda-\mu)r_{ij} & \text{if} \quad i = n, \quad j = m \end{cases} . \quad (46)$$

We have the system of nm equations that has a solution if $\lambda \neq \mu$ thus in the case $\lambda \neq \mu$ we can set B_{11} to zero by adding ΔB_{11}.

Now we consider the case $\lambda = \mu$

$$\Delta B_{11} = -R_{21}^T J_m(\lambda)^T + J_n(\lambda) R_{21}^T$$
$$= \begin{bmatrix} r_{21} - r_{12} & r_{22} - r_{13} & r_{23} - r_{14} & \dots & r_{2m-1} - r_{1m} & r_{2m} \\ r_{31} - r_{22} & r_{32} - r_{23} & r_{33} - r_{24} & \dots & r_{3m-1} - r_{2m} & r_{3m} \\ r_{41} - r_{32} & r_{42} - r_{33} & r_{43} - r_{34} & \dots & r_{4m-1} - r_{3m} & r_{4m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_{n1} - r_{n-12} & r_{n2} - r_{n-13} & r_{n3} - r_{n-14} & \dots & r_{nm-1} - r_{n-1m} & r_{nm} \\ -r_{n2} & -r_{n3} & -r_{n4} & \dots & -r_{nm} & 0 \end{bmatrix} \tag{47}$$

Suppose that $n > m$. We reduce each skew diagonal of B_{11} independently. For the first $m-1$ diagonals we have the system of equations with the matrix (41) that has a solution. For the following $n-m+1$ skew diagonals we have the system with the matrix

$$\begin{pmatrix} 1 & & & & & x_1 \\ -1 & 1 & & & & x_2 \\ & -1 & 1 & & & x_3 \\ & & \ddots & \ddots & & \vdots \\ & & & -1 & 1 & x_k \end{pmatrix} \tag{48}$$

This system has a solution too. But each of the last $m-1$ diagonals has the system of equations with the matrix

$$\begin{pmatrix} -1 & & & & & x_1 \\ 1 & -1 & & & & x_2 \\ & 1 & -1 & & & x_3 \\ & & \ddots & \ddots & & \vdots \\ & & & 1 & -1 & x_{k-1} \\ & & & & 1 & x_k \end{pmatrix} \tag{49}$$

By the Kronecker-Capelli theorem the system does not have a solution. If we throw away the first or the last equation of the system than it will have a solution. This means that we can not set the only one element in each of the last $m-1$ diagonals of B_{11} to zero thus we will have parameters just in the last column or in the last row.

To find solutions for other cases we need to multiply the answer of the first block by $\pm Z$

$$B_{12} - R_{11}^T J_m(\mu) - J_n(\lambda) R_{11}^T$$
$$= B_{11}Z + R_{21}^T ZZJ_m(\mu)^T Z - J_n(\lambda) R_{21}^T Z = \begin{cases} 0Z = 0 & \lambda \neq \mu \\ 0^{\searrow} Z = 0^{\swarrow} & \lambda = \mu \end{cases};$$

$$B_{21} - R_{22}^T J_m(\mu)^T + J_n(\lambda)^T R_{22}^T$$
$$= ZB_{11} + ZR_{21}^T J_m(\mu)^T - ZJ_n(\lambda) ZZR_{21}^T = \begin{cases} Z0 = 0 & \lambda \neq \mu \\ Z0^{\searrow} = 0^{\nearrow} & \lambda = \mu \end{cases};$$

$$B_{22} - R_{12}^T J_m(\mu) - J_n(\lambda)^T R_{12}^T$$
$$= ZB_{11}Z + ZR_{21}^T ZZJ_m(\mu)^T Z - ZJ_n(\lambda) ZZR_{21}^T Z = \begin{cases} Z0Z = 0 & \lambda \neq \mu \\ Z0^{\searrow} Z = 0^{\nwarrow} & \lambda = \mu \end{cases}.$$

Collecting together the answers about all blocks we get that $\mathcal{D}(H_n(\lambda), H_m(\mu))$ is equal to (14) and respectively $\mathcal{D}(K_n, K_m)$ is equal to (15).

3.3.2 Pairs of blocks $\mathcal{D}(L_n, L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to exactly one group of the form (14) by adding

$$(R^T L_m + L_n S, S^T L_n + L_m R), \quad S \in \mathbb{C}^{2n+1\times 2m+1},\ R \in \mathbb{C}^{2m+1\times 2n+1}.$$

Obviously, that we can reduce only on the first pair of matrices, the second pair will be reduced automatically. We reduce a pair of matrices (A,B) by adding

$$\Delta(A,B) = R^T L_m + L_n S$$
$$= (R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} S).$$

It is easily seen that we can set A to zero. To preserve A we must hereafter take R and S such that

$$R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} S = 0.$$

This means

$$\begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} = - \begin{bmatrix} 0 & F_n \\ -F_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

$$\begin{bmatrix} -R_{21}^T F_m^T & R_{11}^T F_m \\ -R_{22}^T F_m^T & R_{12}^T F_m \end{bmatrix} = \begin{bmatrix} -F_n S_{21} & -F_n S_{22} \\ F_n^T S_{11} & F_n^T S_{12} \end{bmatrix}. \tag{50}$$

$B := \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ is reduced by adding

$$\Delta B := \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n \\ -G_n^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$
$$= \begin{bmatrix} -R_{21}^T G_m^T + G_n S_{21} & R_{11}^T G_m + G_n S_{22} \\ -R_{22}^T G_m^T - G_n^T S_{11} & R_{12}^T G_m - G_n^T S_{12} \end{bmatrix},$$

where matrices S_{ij}, R_{ij} , $i,j = 1,2$ satisfy (50).

We reduce each block separately. Firstly we reduce B_{11}. Using the equality $R_{21}^T F_m^T = F_n S_{21}$ we obtain that

$$S_{21} = \begin{bmatrix} & Q & \\ a_1 & \dots & a_m \end{bmatrix}, R_{21}^T = \begin{bmatrix} & b_1 \\ Q & \vdots \\ & b_n \end{bmatrix}, \text{where } Q \text{ is any } n\text{- by-}m \text{ matrix.}$$

Therefore

$$\Delta B_{11} = -R_{21}^T G_m^T + G_n S_{21} = - \begin{bmatrix} & b_1 \\ Q & \vdots \\ & b_n \end{bmatrix} G_m^T + G_n \begin{bmatrix} & Q & \\ a_1 & \dots & a_m \end{bmatrix}$$

$$= \begin{bmatrix} q_{21} - q_{12} & q_{22} - q_{13} & q_{23} - q_{14} & \dots & q_{2m-1} - q_{1m} & q_{2m} - b_1 \\ q_{31} - q_{22} & q_{32} - q_{23} & q_{33} - q_{24} & \dots & q_{3m-1} - q_{2m} & q_{3m} - b_2 \\ q_{41} - q_{32} & q_{42} - q_{33} & q_{43} - q_{34} & \dots & q_{4m-1} - q_{3m} & q_{4m} - b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ q_{n1} - q_{n-12} & q_{n2} - q_{n-13} & q_{n3} - q_{n-14} & \dots & q_{nm-1} - q_{n-1m} & q_{nm} - b_{m-1} \\ a_1 - q_{n2} & a_2 - q_{n3} & a_3 - q_{n4} & \dots & a_{n-1} - q_{nm} & a_n - b_m \end{bmatrix} \tag{51}$$

We can set each skew-diagonal of B_{11} to zero independently by adding corresponding skew-diagonal of ΔB_{11}. Hence we can reduce B_{11} by adding ΔB_{11} to zero.

Now we turn to the second block, that is B_{12} . To preserve A_{12} we take R_{11} and S_{22} such that $R_{11}^T F_m = -F_n S_{22}$ thus

$$S_{22} = \begin{bmatrix} & & & 0 \\ & -R_{11}^T & & \vdots \\ & & & 0 \\ b_1 & \dots & b_m & b_{m+1} \end{bmatrix},$$

where R_{11}^T is any n-by-m matrix. Thus

$$\Delta B_{12} = R_{11}^T G_m + G_n S_{22} = R_{11}^T G_m + G_n \begin{bmatrix} & & & 0 \\ & -R_{11}^T & & \vdots \\ & & & 0 \\ b_1 & \dots & b_m & b_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} -r_{21} & r_{11} - r_{22} & r_{12} - r_{23} & \dots & r_{1m-1} - r_{2m} & r_{1m} \\ -r_{31} & r_{21} - r_{32} & r_{22} - r_{33} & \dots & r_{2m-1} - r_{3m} & r_{2m} \\ -r_{41} & r_{31} - r_{42} & r_{32} - r_{43} & \dots & r_{3m-1} - r_{4m} & r_{3m} \\ -r_{51} & r_{41} - r_{52} & r_{42} - r_{53} & \dots & r_{4m-1} - r_{5m} & r_{4m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -r_{n1} & r_{n-11} - r_{n2} & r_{n-12} - r_{n3} & \dots & r_{n-1m-1} - r_{nm} & r_{n-1m} \\ b_1 & r_{n1} + b_2 & r_{n2} + b_3 & \dots & r_{nm-1} + b_m & r_{nm} + b_{m+1} \end{bmatrix} \tag{52}$$

If $m+1 \geq n$ then we can set B_{12} to zero by adding ΔB_{12}. If $n > m+1$ then we can't set B_{12} to zero. We can set each diagonal to zero independently. By adding the first m diagonals of ΔB_{12} starting from the up right hand corner we set corresponding diagonals of B_{12} to zeros. We can set the next $n-m-1$ diagonals of B_{12} to zeros, except the last element of each of them. We set the last $m+1$ diagonals of B_{12} to zero completely. Hence (A_{12}, B_{12}) is reduced to $(0, \mathcal{Q}_{m+1n}^T)$ by adding $\Delta(A_{12}, B_{12})$.

(A_{21}, B_{21}) is reduced in the same way (up to transposition) as (A_{12}, B_{12}) hence it can be reduced to the form $(0, \mathcal{Q}_{n+1m})$.

Let us look at the last block. Obviously, that one can reduce A_{22} to the form 0_* by adding $\Delta A_{22} = R_{12}^T F_m - F_n^T S_{12}$. To preserve A_{22} we must hereafter

take R_{12} and S_{12} such that $R_{12}^T F_m = F_n^T S_{12}$ thus

$$R_{12}^T = \begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix}, S_{12} = \begin{bmatrix} & & 0 \\ Q & & \vdots \\ & & 0 \end{bmatrix}, \text{ where } Q \text{ is any } n\text{-by-}m \text{ matrix.}$$

Therefore

$$\Delta B_{22} = R_{12}^T G_m - G_n^T S_{12} = \begin{bmatrix} & Q & \\ 0 & \dots & 0 \end{bmatrix} G_m - G_n^T \begin{bmatrix} & & 0 \\ Q & & \vdots \\ & & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & q_{11} & q_{12} & q_{13} & \dots & q_{1m-1} & q_{1m} \\ -q_{11} & q_{21}-q_{12} & q_{22}-q_{13} & q_{23}-q_{14} & \dots & q_{2m-1}-q_{1m} & q_{2m} \\ -q_{21} & q_{31}-q_{22} & q_{32}-q_{23} & q_{33}-q_{24} & \dots & q_{3m-1}-q_{2m} & q_{3m} \\ -q_{31} & q_{41}-q_{32} & q_{42}-q_{33} & q_{43}-q_{34} & \dots & q_{4m-1}-q_{3m} & q_{4m} \\ \hdotsfor{7} \\ -q_{n-11} & q_{n1}-q_{n-12} & q_{n2}-q_{n-13} & q_{n3}-q_{n-14} & \dots & q_{nm-1}-q_{n-1m} & q_{nm} \\ -q_{n1} & -q_{n2} & -q_{n3} & -q_{n4} & \dots & -q_{nm} & 0 \end{bmatrix} \quad (53)$$

By adding ΔB_{22} we can set each element of B_{22} to zero except either the first column and the last row or the first row and the last column.

Collecting together the answers for all blocks we have that $\mathcal{D}(L_n, L_m)$ has the form (16).

3.4 Off-diagonal blocks of matrices of $\mathcal{D}$ that correspond to summands of $(A,B)_{\text{can}}$ of distinct types

Finally, we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(iii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types.

3.4.1 Pairs of blocks $\mathcal{D}(H_n(\lambda), K_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to exactly one group of the form (17) by adding

$$(R^T K_m + H_n(\lambda)S, S^T H_n(\lambda) + K_m R), \quad R \in \mathbb{C}^{2m\times 2n},\ S \in \mathbb{C}^{2n\times 2m}.$$

Obviously, that we can reduce only (A, B) and the second pair will be reduced automatically.

$$\Delta(A,B) = R^T K_m + H_n(\lambda)S =$$
$$(R^T \begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S).$$

It is clear that we can set A to zero. To preserve A we must hereafter take R and S such that

$$R^T \begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0.$$

This means

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & J_m(0) \\ -J_m(0)^T & 0 \end{bmatrix} = \begin{bmatrix} -R_{22}^T J_m(0)^T & R_{12}^T J_m(0) \\ R_{21}^T J_m(0)^T & -R_{11}^T J_m(0) \end{bmatrix}$$

Therefore B is reduced by adding:

$$\Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T J_m(0)^T & R_{12}^T J_m(0) \\ R_{21}^T J_m(0)^T & -R_{11}^T J_m(0) \end{bmatrix}$$
$$= \begin{bmatrix} -R_{21}^T + J_n(\lambda) R_{21}^T J_m(0)^T & R_{11}^T - J_n(\lambda) R_{11}^T J_m(0) \\ -R_{22}^T + J_n(\lambda)^T R_{22}^T J_m(0)^T & R_{12}^T - J_n(\lambda)^T R_{12}^T J_m(0) \end{bmatrix}.$$

The first block B_{11} is reduced by adding

$$\Delta B_{11} = -R_{21}^T + J_n(\lambda) R_{21}^T J_m(0)^T$$
$$= \begin{cases} -r_{ij} + \lambda r_{ij+1} + r_{i+1j+1} & \text{if} \quad 1 \le i \le n-1, \quad 1 \le j \le m-1 \\ -r_{ij} + \lambda r_{ij+1} & \text{if} \quad 1 \le j \le m-1, \quad i = n \\ -r_{ij} & \text{if} \quad 1 \le i \le n, \quad j = m \end{cases}. \tag{54}$$

Adding ΔB_{11} we can set B_{11} to zero (we have a system of nm equations that has a solution).

The reduction of the other blocks follows immediately form the reduction of B_{11} after multiplication by matrices Z (see (38)) of the corresponding size

$R_{11}^T - J_n(\lambda)R_{11}^T J_m(0) = -R_{21}^T Z + J_n(\lambda)R_{21}^T ZZJ_m(0)^T Z,$

$-R_{22}^T + J_n(\lambda)^T R_{22}^T J_m(0)^T = -ZR_{21}^T Z + ZJ_n(\lambda)ZZR_{21}^T ZZJ_m(0)^T Z,$

$R_{12}^T - J_n(\lambda)^T R_{12}^T J_m(0) = -ZR_{21}^T + ZJ_n(\lambda)ZZR_{21}^T J_m(0)^T.$

Collecting together the answers for all blocks we have that $\mathcal{D}(H_n(\lambda), K_m)$ is zero.

3.4.2 Pairs of blocks $\mathcal{D}(H_n(\lambda), L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to zero matrices group by adding

$$(R^T L_m + H_n(\lambda)S, S^T H_n(\lambda) + L_m R), \quad S \in \mathbb{C}^{2n\times 2m+1},\ R \in \mathbb{C}^{2m+1\times 2n}.$$

Obviously, that we can reduce only (A,B) and $(-A^T,-B^T)$ will be reduced automatically.

$$\begin{aligned}
\Delta(A,B) &= R^T L_m + H_n(\lambda)S \\
&= (R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} S).
\end{aligned}$$

It is easy to check that we can set A to zero. To preserve A we must hereafter take R and S such that

$$R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0.$$

This means

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} = \begin{bmatrix} -R_{22}^T F_m^T & R_{12}^T F_m \\ R_{21}^T F_m^T & -R_{11}^T F_m \end{bmatrix}$$

Thus B is reduced by adding

$$\begin{aligned}
\Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} &= \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & J_n(\lambda) \\ -J_n(\lambda)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T F_m^T & R_{12}^T F_m \\ R_{21}^T F_m^T & -R_{11}^T F_m \end{bmatrix} \\
&= \begin{bmatrix} -R_{21}^T G_m^T + J_n(\lambda)R_{21}^T F_m^T & R_{11}^T G_m - J_n(\lambda)R_{11}^T F_m \\ -R_{22}^T G_m^T + J_n(\lambda)^T R_{22}^T F_m^T & R_{12}^T G_m - J_n(\lambda)^T R_{12}^T F_m \end{bmatrix}.
\end{aligned}$$

Let us firstly reduce B_{11}.

$$\Delta B_{11} = -R_{21}^T G_m^T + J_n(\lambda) R_{21}^T F_m^T =$$
$$\begin{bmatrix} -r_{12}+\lambda r_{11}+r_{21} & -r_{13}+\lambda r_{12}+r_{22} & \dots & -r_{1m+1}+\lambda r_{1m}+r_{2m} \\ -r_{22}+\lambda r_{21}+r_{31} & -r_{23}+\lambda r_{22}+r_{32} & \dots & -r_{2m+1}+\lambda r_{2m}+r_{3m} \\ -r_{32}+\lambda r_{31}+r_{41} & -r_{33}+\lambda r_{32}+r_{42} & \dots & -r_{3m+1}+\lambda r_{3m}+r_{4m} \\ \dots & \dots & \dots & \dots \\ -r_{n-12}+\lambda r_{n-11}+r_{n1} & -r_{n-13}+\lambda r_{n-12}+r_{n2} & \dots & -r_{n-1m+1}+\lambda r_{n-1m}+r_{nm} \\ -r_{n2}+\lambda r_{n1} & -r_{n3}+\lambda r_{n2} & \dots & -r_{nm+1}+\lambda r_{nm} \end{bmatrix} \tag{55}$$

Adding this matrix we can set B_{11} to zero. We start to reduce from the n-th row. For it we have the following system of equation with the matrix

$$\begin{pmatrix} \lambda & -1 & & & & x_1 \\ & \lambda & -1 & & & x_2 \\ & & \ddots & \ddots & & \vdots \\ & & & \lambda & -1 & x_k \end{pmatrix}. \tag{56}$$

which has a solution. For $(n-1)$-th row we have

$$\begin{pmatrix} \lambda & -1 & & & 1 & & & y_1 \\ & \lambda & -1 & & & 1 & & y_2 \\ & & \ddots & \ddots & & & \ddots & \vdots \\ & & & \lambda & -1 & & & 1 y_k \end{pmatrix}. \tag{57}$$

But the variables $r_{n1}, r_{n2}, \dots, r_{nm-1}$ are fixed, thus our system becomes like (56), and has a solution. Repeating this reduction to every row from down to up we can set B_{11} to zero.

The block B_{21} is reduced likewise the block B_{11} and thus we omit this verification.

Now we turn to the reduction of B_{12} and B_{22} but it suffice to consider

only B_{12}.

$$\Delta B_{12} = R_{11}^T G_m - J_n(\lambda) R_{11}^T F_m =$$

$$\begin{bmatrix} -\lambda r_{11} - r_{21} & r_{11} - \lambda r_{12} - r_{22} & \dots & r_{1m-1} - \lambda r_{1m} - r_{2m} & r_{1m} \\ -\lambda r_{21} - r_{31} & r_{21} - \lambda r_{22} - r_{32} & \dots & r_{2m-1} - \lambda r_{2m} - r_{3m} & r_{2m} \\ -\lambda r_{31} - r_{41} & r_{31} - \lambda r_{32} - r_{42} & \dots & r_{3m-1} - \lambda r_{3m} - r_{4m} & r_{3m} \\ -\lambda r_{41} - r_{51} & r_{41} - \lambda r_{42} - r_{52} & \dots & r_{4m-1} - \lambda r_{4m} - r_{5m} & r_{4m} \\ -\lambda r_{51} - r_{61} & r_{51} - \lambda r_{52} - r_{62} & \dots & r_{5m-1} - \lambda r_{5m} - r_{6m} & r_{5m} \\ \dots & \dots & \dots & \dots & \dots \\ -\lambda r_{n-11} - r_{n1} & r_{n-11} - \lambda r_{n-12} - r_{n2} & \dots & r_{n-1m-1} - \lambda r_{n-1m} - r_{nm} & r_{n-1m} \\ -\lambda r_{n1} & r_{n1} - \lambda r_{n2} & \dots & r_{nm-1} - \lambda r_{nm} & r_{nm} \end{bmatrix} \tag{58}$$

Adding ΔB_{12} we reduce B_{12} to the form $0^{\leftarrow}$.

Collecting together the answers for all blocks we have that $\mathcal{D}(H_n(\lambda), L_m)$ is equal to (18).

3.4.3 Pairs of blocks $\mathcal{D}(K_n, L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(-A^T,-B^T))$ can be reduced to zero matrices group by adding

$$(R^T L_m + K_n S, S^T K_n + L_m R), \quad S \in \mathbb{C}^{2n \times 2m+1},\ R \in \mathbb{C}^{2m+1 \times 2n}.$$

Obviously, that we can look only on the first pair of matrices, the second pair will be reduced automatically.

$$\Delta(A,B) = R^T L_m + K_n S$$
$$= (R^T \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S).$$

It is clear that we can set B to zero. To preserve B we must hereafter take R and S such that

$$R^T \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S = 0.$$

This means

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} = \begin{bmatrix} -R_{22}^T G_m^T & R_{12}^T G_m \\ R_{21}^T G_m^T & -R_{11}^T G_m \end{bmatrix}$$

Hence A is reduced by adding

$$\Delta A = \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \begin{bmatrix} -R_{22}^T G_m^T & R_{12}^T G_m \\ R_{21}^T G_m^T & -R_{11}^T G_m \end{bmatrix}$$
$$= \begin{bmatrix} -R_{21}^T F_m^T + J_n(0) R_{21}^T G_m^T & R_{11}^T F_m - J_n(0) R_{11}^T G_m \\ -R_{22}^T F_m^T + J_n(0)^T R_{22}^T G_m^T & R_{12}^T F_m - J_n(0)^T R_{12}^T G_m \end{bmatrix}.$$

Let us reduce only on the block A_{11} and the block A_{21} is reduced analogically.

$$\Delta A_{11} = -R_{21}^T F_m^T + J_n(0) R_{21}^T G_m^T =$$
$$\begin{bmatrix} -r_{11} + r_{22} & -r_{12} + r_{23} & -r_{13} + r_{24} & \dots & -r_{1m} + r_{2m+1} \\ -r_{21} + r_{32} & -r_{22} + r_{33} & -r_{23} + r_{34} & \dots & -r_{2m} + r_{3m+1} \\ -r_{31} + r_{42} & -r_{32} + r_{43} & -r_{33} + r_{44} & \dots & -r_{3m} + r_{4m+1} \\ \dots & \dots & \dots & \dots & \dots \\ -r_{n-11} + r_{n2} & -r_{n-12} + r_{n3} & -r_{n-13} + r_{n4} & \dots & -r_{n-1m} + r_{nm+1} \\ -r_{n1} & -r_{n2} & -r_{n4} & \dots & -r_{nm} \end{bmatrix} \tag{59}$$

Adding ΔA_{11} we set A_{11} to zero. We reduce every diagonal of A_{11} independently. For each of the first m diagonals we have the system of equations with the matrix

$$\begin{pmatrix} -1 & 1 & & & & x_1 \\ & -1 & 1 & & & x_2 \\ & & \ddots & \ddots & & \vdots \\ & & & -1 & 1 & x_{k-1} \\ & & & & -1 & x_k \end{pmatrix} \tag{60}$$

which has a solution. And for the others we have

$$\begin{pmatrix} -1 & 1 & & & & x_1 \\ & -1 & 1 & & & x_2 \\ & & \ddots & \ddots & & \vdots \\ & & & -1 & 1 & x_k \end{pmatrix} \tag{61}$$

which has a solution too. Thus we can set A_{11} to zero.

Hence, we turn to the reduction of the blocks A_{12} and A_{22} but it is enough

to consider only A_{12}.

$$\Delta A_{12} = R_{11}^T F_m - J_n(0) R_{11}^T G_m =$$

$$\begin{bmatrix} r_{11} & r_{12} - r_{21} & r_{13} - r_{22} & \dots & r_{1m} - r_{2m-1} & -r_{2m} \\ r_{21} & r_{22} - r_{31} & r_{13} - r_{32} & \dots & r_{2m} - r_{3m-1} & -r_{2m} \\ r_{31} & r_{32} - r_{41} & r_{13} - r_{42} & \dots & r_{3m} - r_{4m-1} & -r_{3m} \\ r_{41} & r_{42} - r_{51} & r_{13} - r_{52} & \dots & r_{4m} - r_{5m-1} & -r_{4m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r_{n-11} & r_{n-12} - r_{n1} & r_{n-13} - r_{n2} & \dots & r_{n-1m} - r_{nm-1} & -r_{nm} \\ r_{n1} & r_{n2} & r_{n3} & \dots & r_{nm} & 0 \end{bmatrix} \tag{62}$$

Adding ΔA_{12} we reduce A_{12} to the form $0^{\rightarrow}$.

Collecting the answers for all block we have $\mathcal{D}(K_n, L_m)$ is equal to (19).

4 Versality in Lemma 3.2

In this section we give a constructive proof of the versality of each deformation $(A, B) + \mathcal{D}(\vec{\varepsilon})$, in which $\mathcal{D}$ satisfies (27): in Lemma 4.3 we construct a deformation $\mathcal{I}(\vec{\varepsilon})$ of the identity matrix such that

$$(A, B) + \mathcal{D}(\vec{\varepsilon}) = \mathcal{I}(\vec{\varepsilon})^T \mathcal{U}(\vec{\varepsilon}) \mathcal{I}(\vec{\varepsilon}), \tag{63}$$

where $\mathcal{U}(\vec{\varepsilon})$ is defined in (25).

We use the Frobenius norm of a complex $n \times n$ matrix $P = [p_{ij}]$:

$$\|P\| := \sqrt{\sum |p_{ij}|^2},$$

and for each of $(0,\ast)$ matrix of $\mathcal{D}$ of the same size $n \times n$, we write

$$\|P\|_{\mathcal{D}} := \sqrt{\sum_{(i,j) \notin \mathcal{I}(\mathcal{D})} |p_{ij}|^2},$$

where $\mathcal{I}(\mathcal{D})$ is the set (24) of indices of the stars in $\mathcal{D}$.

By [5, Section 5.6],

$$\|aP + bQ\| \leqslant |a| \, \|P\| + |b| \, \|Q\|, \qquad \|PQ\| \leqslant \|P\| \, \|Q\| \tag{64}$$

for matrices P and Q and $a, b \in \mathbb{C}$.

Lemma 4.1. *Let* $(A,B) \in (\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ *and let* $\mathcal{D}$ *be a* $(0,*)$ *matrix of size* $n\times n$ *satisfying*

$$(\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n}) = T(A,B) + \mathcal{D}(\mathbb{C}). \tag{65}$$

Then there exists a natural number m *such that for each real numbers* ε *and* δ *satisfying*

$$0 < \varepsilon \leqslant \delta < \frac{1}{m} \tag{66}$$

and for each n*-by-*n *matrices* M *and* R *satisfying*

$$\|M\|_{\mathcal{D}} < \varepsilon, \qquad \|R\|_{\mathcal{D}} < \varepsilon, \qquad \|M\| < \delta, \qquad \|R\| < \delta \tag{67}$$

there exists

$$S = I_n + X, \qquad \|X\| < m\varepsilon, \tag{68}$$

in which the entries of X *are linear polynomials in entries of* M *and* R *such that*

$$S^T(A+M, B+R)S = (A+N, B+Q), \tag{69}$$

$$\|N\|_{\mathcal{D}} < m\varepsilon^2, \quad \|Q\|_{\mathcal{D}} < m\varepsilon^2, \quad \|N\| < \delta + m\varepsilon, \quad \|Q\| < \delta + m\varepsilon. \tag{70}$$

Proof. First we construct $S = I_n + X$. By (65), for each n-by-n pairs of matrices $(E_{ij}, 0)$ and $(0, F_{ij})$ there exists $X_{ij}, X'_{ij} \in \mathbb{C}^{n\times n}$ such that

$$(E_{ij}, 0) + X_{ij}^T(A+M, B+R) + (A+M, B+R)X_{ij} \in \mathcal{D}(\mathbb{C})$$

$$(0, F_{ij}) + X_{ij}'^T(A+M, B+R) + (A+M, B+R)X'_{ij} \in \mathcal{D}(\mathbb{C})$$

where $\mathcal{D}(\mathbb{C})$ is defined in (22). If $M = \sum_{i,j} m_{ij}E_{ij}, R = \sum_{i,j} r_{ij}F_{ij}$ (it means that $M = [m_{ij}], R = [r_{ij}]$), then

$$\sum_{i,j}(m_{ij}E_{ij}, r_{ij}F_{ij}) + \sum_{i,j}(m_{ij}X_{ij}^T + r_{ij}X_{ij}'^T)(A+M, B+R) \\ + (A+M, B+R)\sum_{i,j}(m_{ij}X_{ij} + r_{ij}X'_{ij}) \in \mathcal{D}(\mathbb{C})$$

and for

$$X := \sum_{i,j}(m_{ij}X_{ij} + r_{ij}X'_{ij})$$

we have

$$(M,R) + X^T(A+M, B+R) + (A+M, B+R)X \in \mathcal{D}(\mathbb{C}). \tag{71}$$

If $(i,j) \in \mathcal{I}(\mathcal{D})$, then $(E_{ij},0) \in \mathcal{D}(\mathbb{C})$, and so we can put $X_{ij} = 0$. If $(i,j) \notin \mathcal{I}(\mathcal{D})$, then $|m_{ij}| < \varepsilon$ by the first inequality in (67), analogously for the second matrix thus $|r_{ij}| < \varepsilon$. We obtain

$$\begin{aligned}\|X\| &\leqslant \sum_{(i,j)\notin\mathcal{I}_1(\mathcal{D})} |m_{ij}|\|X_{ij}\| + \sum_{(i,j)\notin\mathcal{I}_2(\mathcal{D})} |r_{ij}|\|X'_{ij}\| \\ &< \sum_{(i,j)\notin\mathcal{I}_1(\mathcal{D})} \varepsilon\|X_{ij}\| + \sum_{(i,j)\notin\mathcal{I}_2(\mathcal{D})} \varepsilon\|X'_{ij}\| = \varepsilon c,\end{aligned}$$

where

$$c := \sum_{(i,j)\notin\mathcal{I}_1(\mathcal{D})} \|X_{ij}\| + \sum_{(i,j)\notin\mathcal{I}_2(\mathcal{D})} \|X'_{ij}\|.$$

Put

$$S^T(A+M, B+R)S = (A+N, B+Q) \qquad S := I_n + X,$$

then

$$\begin{aligned}(N,Q) = (M,N) + X^T(A+M,B+R) + (A+M,B+R)X \\ +X^T(A+M,B+R)X.\end{aligned} \tag{72}$$

Denote $a := \|A\|$, then

$$\begin{aligned}\|N\| &\leqslant \|M\| + 2\|X\|(\|A\| + \|M\|) + \|X\|^2\|A+M\| \\ &< \delta + 2\varepsilon c(a+\delta) + \varepsilon^2c^2(a+\delta) = \delta + \varepsilon c(a+\delta)(2+\varepsilon c) \\ &< \delta + \varepsilon c(a+1)(2+c).\end{aligned}$$

Analogously $b := \|B\|$, then

$$\begin{aligned}\|Q\| &\leqslant \|R\| + 2\|X\|(\|B\| + \|R\|) + \|X\|^2\|B+R\| \\ &< \delta + \varepsilon c(b+1)(2+c).\end{aligned}$$

By (71) and (72),

$$\begin{aligned}\|N\|_{\mathcal{D}} &= \|X^T(A+M)X\| \leqslant \|X\|^2(\|A\| + \|M\|) \\ &< (\varepsilon c)^2(a+\delta) < \varepsilon^2c^2(a+1), \\ \|Q\|_{\mathcal{D}} &= \|X^T(B+R)X\| \leqslant \|X\|^2(\|B\| + \|R\|) \\ &< (\varepsilon c)^2(b+\delta) < \varepsilon^2c^2(b+1).\end{aligned}$$

Any natural number m that is greater than c, $c(a+1)(2+c)$, $c(b+1)(2+c)$,$c^2(a+1)$ and $c^2(b+1)$ ensures the equalities in (68) and (69). $\square$

Lemma 4.2. *[4] Let m be any natural number being $\geqslant 3$, and let*

$$\varepsilon_1,\ \delta_1,\ \varepsilon_2,\ \delta_2,\ \varepsilon_3,\ \delta_3,\ \ldots$$

be the sequence of numbers defined by induction:

$$\varepsilon_1 = \delta_1 = m^{-4}, \qquad \varepsilon_{i+1} = m\varepsilon_i^2, \qquad \delta_{i+1} = \delta_i + m\varepsilon_i. \tag{73}$$

Then

$$\varepsilon_i \leqslant m^{-2i}, \qquad \delta_i < m^{-2} \tag{74}$$

for all i, and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots < 1. \tag{75}$$

Proof. The first inequality in (74) holds for $i = 1$ and $i = 2$ since

$$\varepsilon_1 = m^{-4} < m^{-2}, \qquad \varepsilon_2 = m\varepsilon_1^2 = mm^{-8} = m^{-7} < m^{-4}.$$

Reasoning by induction, we assume that it holds for some $i \geqslant 2$, then

$$\varepsilon_{i+1} = m\varepsilon_i^2 \leqslant mm^{-4i} = m^{-2(i+1)}m^{3-2i} < m^{-2(i+1)}.$$

This proves the first inequality in (74). Then (75) holds too since

$$\begin{aligned} \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots &\leqslant m^{-4} + m^{-7} + m^{-6} + m^{-8} + \cdots \\ &< m^{-4}(1 + m^{-1} + m^{-2} + m^{-3} + \cdots) \\ &= m^{-4}\frac{1}{1 - m^{-1}} \leqslant \frac{3}{2}m^{-4}. \end{aligned}$$

The second inequality in (74) holds for all i since

$$\begin{aligned} \delta_i &= \delta_{i-1} + m\varepsilon_{i-1} = \delta_{i-2} + m(\varepsilon_{i-2} + \varepsilon_{i-1}) = \cdots \\ &= \delta_1 + m(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{i-1}) \\ &< m^{-4} + \frac{3}{2}m^{-3} \leqslant m^{-3}\left(m^{-1} + \frac{3}{2}\right) < 3m^{-3} \leqslant m^{-2}. \end{aligned}$$

□

The versality of each deformation $(A, B) + \mathcal{D}(\vec{\varepsilon})$, in which $\mathcal{D}$ satisfies (27), follows from the following lemma.

Lemma 4.3. *Let $(A,B) \in (\mathbb{C}_c^{n\times n}, \mathbb{C}_c^{n\times n})$ and let $\mathcal{D}$ be a pair $(0,*)$ matrices of size $n\times n$ satisfying* (65)*. Let m be a natural number that is greater or equal than* 3 *and satisfies Lemma 4.1, and let M and R are any n-by-n matrices, such that $\|M\| < m^{-4}$, $\|R\| < m^{-4}$. Then there exists a matrix $S = I_n + X$ depending holomorphically on the entries of M and R in a neighborhood of zero such that*

$$S^T(A+M, B+R)S - (A,B) \in \mathcal{D}(\mathbb{C})$$

and $S = I_n$ if $(M,R) = (0,0)$.

Proof. We construct a sequence of matrices

$$(A,B) + (M_1,R_1), \quad (A,B) + (M_2,R_2), \quad (A,B) + (M_3,R_3), \ \dots$$

by induction. Put $(M_1,R_1) = (M,R)$. Let (M_i,R_i) be constructed and let

$$\|M_i\|_{\mathcal{D}} < \varepsilon_i, \qquad \|R_i\|_{\mathcal{D}} < \varepsilon_i, \qquad \|M_i\| < \delta_i, \qquad \|R_i\| < \delta_i,$$

where ε_i and δ_i are defined in (73). Then by (74) and Lemma 4.1 there exists

$$S_i = I_n + X_i, \qquad \|X_i\| < m\varepsilon_{i+1}, \tag{76}$$

such that

$$S_i^T(A+M_i, B+R_i)S_i = (A+M_{i+1}, B+R_{i+1}),$$

$$\|M_{i+1}\|_{\mathcal{D}} < \varepsilon_{i+1}, \quad \|R_{i+1}\|_{\mathcal{D}} < \varepsilon_{i+1}, \quad \|M_{i+1}\| < \delta_{i+1}, \quad \|R_{i+1}\| < \delta_{i+1}.$$

For each natural number l, put

$$S^{(l)} := S_1 S_2 \cdots S_l = (I_n + X_1)(I_n + X_2)\cdots(I_n + X_l). \tag{77}$$

Now let $l \to \infty$ then

$$\lim_{l\to\infty} S^{(l)} = S_1 S_2 \cdot \ldots \cdot S_l \cdot \ldots = \prod_{i=1}^{\infty}(I_n + X_i). \tag{78}$$

The sum

$$\|X_1\| + \|X_2\| + \|X_3\| + \cdots = \sum_{i=1}^{\infty} \|X_i\|$$

absolutely converges due to (76) and (75) thus by [7, Theorem 4] the product (78) converges to some invertible matrix S. The entries of S are holomorphic functions in the entries of M (that satisfies $\|M\| < m^{-4}$).

Since $(A+M_l, B+R_l) \to S^T(A+M, B+R)S$ if $l \to \infty$, $\|M_l\|_{\mathcal{D}} < \varepsilon_l \to 0$ and $\|R_l\|_{\mathcal{D}} < \varepsilon_l \to 0$, we have $S^T(A+M, B+R)S - (A,B) \in \mathcal{D}(\mathbb{C})$. ∎

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